独立な正規分布の合成

$X \sim N(0, \sigma_1{}^2)$, $Y \sim N(0, \sigma_2{}^2)$, X と Y は互いに独立であるとする．
$U = X + Y$, $V = X - Y$ とおくとき，次の問いに答えよ．

(1) $V[U]$ と $V[V]$ を求めよ．

(2) $Cov[U, V]$ を求めよ．

(3) U と V の相関係数を求めよ．

ポイント

分散の性質

$$V[aX + bY + c] = a^2 V[X] + b^2 V[Y]$$

共分散の定義

$$Cov[X, Y] := E[(X - E(X))(Y - E(Y))]$$

相関係数の定義

$$\rho[X, Y] := \frac{Cov[X, Y]}{\sqrt{V[X]}\sqrt{V[Y]}}$$

(1) $V[U]$

$$
\begin{aligned}
&= V[X + Y] \\
&= V[X] + V[Y] \\
&= \underline{\sigma_1^2 + \sigma_2^2} \quad /\!/
\end{aligned}
$$

$V[V]$

$$
\begin{aligned}
&= V[X - Y] \\
&= V[X] + (-1)^2 V[Y] \\
&= V[X] + V[Y] \\
&= \underline{\sigma_1^2 + \sigma_2^2} \quad /\!/
\end{aligned}
$$

(2) $Cov[U, V]$

$$
\begin{aligned}
&= Cov[X + Y, X - Y] \\
&= E[(X + Y - E[X + Y]) \cdot (X - Y - E[X - Y])] \\
&= E[(\underline{X - E[X]} + \underline{Y - E[Y]}) \cdot (\underline{X - E[X]} - (\underline{Y - E[Y]}))] \\
&= E[(X - E[X])^2 - (Y - E[Y])^2] \\
&= E[V[X] - V[Y]] \\
&= V[X] - V[Y] = \underline{\sigma_1^2 - \sigma_2^2} \quad /\!/
\end{aligned}
$$

(3)

$$
\begin{aligned}
\rho[U, V] &= \frac{Cov[U, V]}{\sqrt{V[U]}\sqrt{V[V]}} \\
&= \frac{\sigma_1^2 - \sigma_2^2}{\sqrt{\sigma_1^2 + \sigma_2^2}\sqrt{\sigma_1^2 + \sigma_2^2}} \\
&= \underline{\frac{\sigma_1^2 - \sigma_2^2}{\sigma_1^2 + \sigma_2^2}} \quad /\!/
\end{aligned}
$$

$U \sim N(0, \sigma_1^2 + \sigma_2^2)$

$V \sim N(0, \sigma_1^2 + \sigma_2^2)$